Written Exam Advanced Mechanics

Wednesday January 20, 2021

Indicate at the first page clearly your name and student number.

Problem 1: A rigid three-particle system (10 pts)

Consider a rigid three-particle system of masses m_1, m_2, m_3 with coordinates (x, y, z) as follows:

$$\begin{aligned} m_1 &= 3m\,, \quad (x, y, z) = (b, 0, b)\,, \\ m_2 &= 4m\,, \quad (x, y, z) = (b, b, -b)\,, \\ m_3 &= 2m\,, \quad (x, y, z) = (-b, b, 0) \end{aligned} \tag{1}$$

for some constant b. The general form for the inertia tensor I_{ij} is given by

$$I_{ij} = \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j} \right) . \tag{2}$$

a. Explain the meaning of the indices i and α. Show that $I_{ij} = I_{ji}$. *(2 points)*

b. The inertia tensor for this system is given by (using matrix notation)

$$I = mb^2 \begin{pmatrix} I_{11} & I_{12} & 1 \\ I_{21} & 16 & 4 \\ 1 & 4 & 15 \end{pmatrix} . \tag{3}$$

Calculate the missing expressions for I_{11} and $I_{12} = I_{21}$ and show that they are given by $I_{11} = 13mb^2$ and $I_{12} = I_{21} = -2mb^2$. *(2 points)*

c. Show that one of the principle moments of inertia is given by $I_1 = 10mb^2$. *(2 points)*

d. Calculate the angular momentum vector $\vec{L}$ for the case that the system rotates around the z-axis, i.e. $\vec{\omega} = (0, 0, \omega_z)$. Is $\vec{L}$ in the same direction as $\vec{\omega}$? *(2 points)*

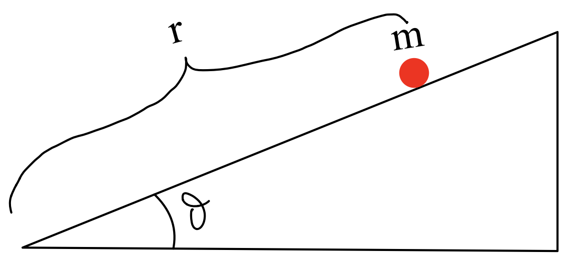
e. Calculate the components of the principal axis $\vec{\mathrm{p}}_1$ (normalized to have length one)

$$\vec{\mathrm{p}}_1 = \begin{pmatrix} \mathrm{p}_{11} \\ \mathrm{p}_{21} \\ \mathrm{p}_{31} \end{pmatrix} \tag{4}$$

of the inertia tensor (3) corresponding to $I_1 = 10mb^2$. *(2 points)*

Problem 2: Particle on a slope (10 pts)

Consider a particle of mass m that rests on a smooth plane. The plane is raised to an inclination angle θ at a rate $\dot{\theta} = \alpha$ for constant α, causing the particle to move down the plane. Put the origin of the reference frame at the bottom of the slope. Call the distance from the bottom of the slope to the particle r, see the figure below. We assume that $\theta = 0$ at $t = 0$ and that there is no friction.



a. Indicate the generalized coordinate(s) you will be using to describe this problem and explain your answer. *(1 point)*

b. Calculate the kinetic energy T and potential energy U of the particle and give the Lagrangian L. Take $U = 0$ at the bottom of the slope. *(2 points)*

c. Show that the Euler-Lagrange equations of motion are given by

$$\ddot{r} - \alpha^2 r = -g \sin \alpha t \,, \tag{5}$$

where g is the gravitational acceleration constant. *(2 points)*

d. Give the general solution $r = r_h + r_p$ of the equation of motion (5) where r_h is the general solution of the homogeneous equation $\ddot{r} - \alpha^2 r = 0$ and r_p

is a particular solution of the inhomogeneous equation (5). Assume that the initial conditions are given by

$$r(0) = r_0 \qquad \text{and} \qquad \dot{r}(0) = 0\,. \tag{6}$$

Hint: Parametrize the particular solution r_p as

$$r_p = C \sin \alpha t \tag{7}$$

for some constant C. *(3 points)*

e. Show that in the general solution $r = r_h + r_p$ the terms of order α^{-1} cancel. Take the limit that α goes to zero. Explain why the resulting solution is expected. *(2 points)*

Problem 3: The Electromagnetic Field (10 pts)

Consider the Lagrangian density $\mathcal{L}$ for the electromagnetic potential field $A^\mu(x)$ coupled to a four-current $j_\mu(x)$

$$\mathcal{L} = -\frac{1}{16\pi c} F^{\mu\nu} F_{\mu\nu} - \frac{1}{c^2} A^\mu j_\mu\,, \tag{8}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$.

a. Show that $\tilde{F}^{\mu\nu} = F^{\mu\nu}$ for a gauge transformation $\tilde{A}^\mu(x) = A^\mu(x) - \partial^\mu \chi(x)$ with arbitrary function $\chi(x)$. *(1 point)*

The Euler-Lagrange equation for fields is given by

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) = 0\,. \tag{9}$$

b. Calculate $\frac{\partial \mathcal{L}}{\partial A_\nu}$ and $\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}$ and show that the equation of motion for A_ν is given by *(3 points)*

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\nu\,. \tag{10}$$

c. Show that in the Lorentz gauge $\partial_\mu A^\mu = 0$ the equation of motion (10) reduces to the wave equation

$$\Box A^\mu = \frac{4\pi}{c} j^\mu \,. \tag{11}$$

with $\Box = \partial^\mu \partial_\mu$. (*1 point)*

We write the electromagnetic potential field A^μ in terms of a constant polarization vector S^μ and a constant wave vector k^μ as follows:

$$A^\mu = S^\mu e^{ik\cdot x} \tag{12}$$

with $k^\mu = (\omega/c, \vec{k})$ and $k \cdot x = k^\mu x_\mu$.

d. Take $j^\mu = 0$. Show that $A^\mu(x)$ satisfies the equation of motion (11), with $j^\mu = 0$, provided that $k^2 = 0$. Show that $A^\mu(x)$ also satisfies the Lorentz gauge provided that $k_\mu S^\mu = 0$. *(2 points)*

e. Consider the following wave vector k^μ and polarization vector S^μ of an electromagnetic potential field A^μ that satisfies $k^2 = k_\mu S^\mu = 0$:

$$k^\mu = (\omega/c, 0, 0, \omega/c) \quad \text{and} \quad S^\mu = (0, 0, 1, 0)\,. \tag{13}$$

Calculate the electromagnetic tensor field $F^{\mu\nu}$ and the corresponding electric field $\vec{E}$ and magnetic field $\vec{B}$. What are the directions of the electric and magnetic fields and in which direction is the wave traveling? *(3 points)*